French Baccalaureate Oral Examination

Why Are Logarithms Essential in Finance?

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Topic: The Natural Logarithm and Its Applications in Economics

Introduction

When we first encounter the natural logarithm in our final year of secondary school, it is easy to see it as a purely abstract mathematical object: the inverse of the exponential function, useful for solving theoretical equations. Yet as soon as we turn our attention to economics and financial markets, the logarithm emerges as a tool that is both widespread and fundamental. It plays a central role whenever we need to analyse growth, model interest rates or measure the performance of a portfolio of assets.

Consider a simple financial example. A saver invests an initial sum at a fixed annual interest rate of 5%. By multiplying the balance by the same factor each year, it is easy to calculate how much this investment will be worth after one year or after two. But a more strategic question arises when we look at the same investment from the other direction: *how long will it take for the initial investment to double?* The unknown is no longer a factor that we multiply by. It appears in the exponent itself. This is precisely where ordinary arithmetic is no longer enough and the logarithm becomes indispensable.

This brings us to the central question: **what makes the natural logarithm an indispensable conceptual and mathematical tool for modelling and understanding how financial mechanisms work?**

My argument is that the logarithm is essential because it makes a fundamental mathematical transformation possible. It converts multiplicative growth, which is often difficult to interpret and work with over several periods, into a linear, additive measure that is much easier to handle. To show how this works, I will examine three of its main applications: solving for time in equations for compound growth, expressing continuous compounding mathematically and turning compounded financial returns into sums.

1 Solving for time under compound growth

Under compound interest, an investment or a debt grows exponentially rather than linearly. Its value increases by a multiplicative factor each period, rather than by a fixed amount.

1.1 Modelling growth under discrete compounding

Suppose an initial sum C_0 is invested at an annual interest rate i, expressed as a decimal (for example, 0.05 for 5%). Each year's interest is added to the principal and itself earns interest the following year. After n years, the balance C_n is given by the general term of a geometric sequence with common ratio $(1 + i)$:

$$C_n = C_0 \times (1 + i)^n$$

It is straightforward to calculate C_n when n is known. To find the time n needed to reach a particular financial target, however, we need the logarithm.

1.2 Solving algebraically for the time it takes to double

Let us return to our example. We want to find the time n it takes for an initial investment of 1,000 euros to double to 2,000 euros at an annual interest rate of 5% ($i = 0.05$). The equation is

$$1{,}000 \times (1.05)^n = 2{,}000$$

Dividing both sides by 1,000 leaves the power on its own:

$$(1.05)^n = 2$$

Since the unknown n is in the exponent, we take the natural logarithm, ln, of both sides. This function is strictly increasing on $(0, +\infty)$:

$$\ln((1.05)^n) = \ln(2)$$

The power rule for logarithms says that $\ln(a^n) = n \ln(a)$ for every $a > 0$ and $n \in \mathbb{N}$. Applying this property lets us bring the exponent down and write it as a multiplicative factor:

$$n \times \ln(1.05) = \ln(2)$$

Since $\ln(1.05) \neq 0$, we can divide both sides to solve for n:

$$n = \frac{\ln(2)}{\ln(1.05)}$$

Substituting the approximate values used in class, $\ln(2) \approx 0.6931$ and $\ln(1.05) \approx 0.0488$, gives

$$n \approx \frac{0.6931}{0.0488} \approx 14.2 \text{ years}$$

1.3 A financial application: the Rule of 72

This calculation is the basis of a well-known shortcut in finance: the **Rule of 72**. For small values of i, the first-order expansion of the natural logarithm about 1 gives $\ln(1 + i) \approx i$. We can therefore approximate the time it takes for the initial investment to double:

$$n = \frac{\ln(2)}{\ln(1 + i)} \approx \frac{0.693}{i}$$

If the interest rate is expressed as a percentage, $t = 100 \times i$, the formula becomes $n \approx \frac{69.3}{t}$. For mental arithmetic, practitioners use 72 in place of 69.3 because it has many convenient divisors. This gives the rule of thumb doubling time $\approx 72/t$. In our example, $72/5 = 14.4$ years, very close to the value obtained using logarithms.

2 Continuous compounding and the exponential function

In practice, financial flows do not stop on 31 December. Interest may be credited monthly or daily, and it can even be modelled as accruing continuously, at every instant.

2.1 Taking the limit as compounding becomes continuous

If interest at an annual rate r is compounded m times a year, the balance after t years is $C_0\left(1+\frac{r}{m}\right)^{mt}$. As the compounding frequency increases without bound ($m \to +\infty$), the intervals between interest payments shrink to zero. This is continuous compounding. Using the limit studied in our final-year mathematics course, we obtain

$$\lim_{m\to+\infty} C_0\left(1+\frac{r}{m}\right)^{mt} = C_0\, e^{rt}$$

The continuous compounding model therefore takes the form of an exponential function:

$$C(t) = C_0\, e^{rt}$$

2.2 Finding the instantaneous growth rate

The natural logarithm is the inverse of the exponential function: for every $x \in \mathbb{R}$, $\ln(e^x) = x$. It therefore lets us undo exponentiation and recover the interest rate applied over a period when we know the initial and final balances. We start from the continuous compounding relationship:

$$\frac{C(t)}{C_0} = e^{rt}$$

Taking the natural logarithm of both sides lets us recover the exponent:

$$\ln\left(\frac{C(t)}{C_0}\right) = \ln\left(e^{rt}\right)$$

$$\ln\left(\frac{C(t)}{C_0}\right) = rt$$

We can now solve for the continuously compounded interest rate r:

$$r = \frac{1}{t}\ln\left(\frac{C(t)}{C_0}\right)$$

This formula gives the natural logarithm a direct financial interpretation. The logarithm of the ratio of final to initial wealth measures continuously compounded growth over the period. Dividing by the length of the period gives the instantaneous growth rate of the capital.

3 Why logarithmic returns add up over time

In financial markets, analysts often prefer to work with logarithmic returns, also called *log returns*, rather than ordinary percentage returns. The reason lies in a fundamental algebraic property of the logarithm: it turns the product of a series of factors into a sum.

3.1 The pitfall of adding simple returns

The simple return on an asset between two dates is $R = \frac{P_{\text{final}} - P_{\text{initial}}}{P_{\text{initial}}}$. Consider an asset initially worth 100 euros.

- **Period 1:** The asset rises from 100 to 120 euros. Its simple return is $+20\%$.
- **Period 2:** The asset then falls from 120 to 90 euros. Its simple return is $\frac{90-120}{120} = -25\%$.

An analyst who simply takes the arithmetic mean of these two returns obtains

$$\frac{20\% + (-25\%)}{2} = -2.5\% \text{ per period}$$

Yet the asset has fallen from 100 to 90 euros, an actual loss of 10% over the two periods. Applying a return of -2.5% twice does not give that result. Simple returns compound through multiplication of their growth factors:

$$100 \times (1 + 0.20) \times (1 - 0.25) = 100 \times 1.20 \times 0.75 = 90$$

3.2 Turning compounded returns into sums

To account for compounding and make returns easier to analyse statistically, quantitative finance uses the logarithmic return, defined as

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

Let us calculate the logarithmic returns in our example:

- **Period 1:** $r_1 = \ln\left(\frac{120}{100}\right) \approx 0.1823$ (or $+18.23\%$)
- **Period 2:** $r_2 = \ln\left(\frac{90}{120}\right) \approx -0.2877$ (or -28.77%)
- **Sum of the returns:** $r_{\text{total}} = r_1 + r_2 = \ln\left(\frac{120}{100}\right) + \ln\left(\frac{90}{120}\right)$

The product rule for logarithms, $\ln(a) + \ln(b) = \ln(ab)$, lets us simplify the sum of the two returns:

$$r_{\text{total}} = \ln\left(\frac{120}{100} \times \frac{90}{120}\right) = \ln\left(\frac{90}{100}\right) = \ln(0.9) \approx -0.1054 \quad (\text{or } -10.54\%)$$

The logarithmic return over the whole period is exactly equal to the sum of the logarithmic returns for the individual periods. Multiplying successive growth factors has become a matter of addition. This property, known as time additivity, makes financial time series easier to work with mathematically.

Limitations of logarithms in finance

For all its power, the logarithm has limits as a financial tool. Three qualifications are particularly important when interpreting the results.

1. **Restrictions on the domain:** The function $\ln(x)$ is defined only for strictly positive values of x. This works well for assets with positive prices. The same approach cannot be applied directly to a portfolio whose value falls to zero or becomes negative, or to a series of cash flows that alternate between gains and losses.

2. **Less intuitive for investors:** A logarithmic return is an abstract measure. Telling retail investors that a fund earned a continuously compounded return of 9.53% is less intuitive than saying it gained 10%. Over one year, the two figures describe the same change in value, to the precision shown.

3. **No predictive power:** The logarithm is a tool for measurement and modelling. It cannot predict where markets will go next or eliminate the risk of a stock market crash.

Conclusion

The natural logarithm is much more than a theoretical exercise in the final-year mathematics curriculum. It is one of the mathematical foundations of modern finance for three main reasons:

- It lets us **solve for time** in the exponent of a compound interest calculation.
- It allows us to **solve for the interest rate** from the initial and final balances under continuous compounding.
- It turns the multiplication of successive growth factors into a sum, giving us a **linear, additive framework** that is easier to work with.

Without the logarithm, financial modelling would be tied to cumbersome calculations involving products. Turning those products into sums makes complex changes easier to understand. In finance, a clear and rigorous measure is the starting point for an informed investment decision.

Proofs

A.1 Deriving the continuous compounding formula

Question *Can you show, step by step, how taking a limit gives the formula for continuous compounding?*

Detailed answer In the discrete compounding model, we invest an initial sum C_0 at a nominal annual interest rate r, with interest compounded m times a year. A geometric sequence gives the balance after t years as

$$A_m = C_0 \left(1 + \frac{r}{m}\right)^{mt}.$$

We want to find the limit of A_m as the compounding frequency m tends to infinity. If $r = 0$, the balance stays at C_0 and the result follows directly. For $r \neq 0$, m appears in both the base and the exponent, giving the indeterminate form 1^∞.

We bring the exponent down by taking the natural logarithm of the ratio A_m/C_0.

$$\ln\left(\frac{A_m}{C_0}\right) = \ln\left(\left(1 + \frac{r}{m}\right)^{mt}\right) = mt \ln\left(1 + \frac{r}{m}\right).$$

For a fixed r, set $x = \frac{r}{m}$. Then $x \to 0$ as $m \to +\infty$, and substituting $m = \frac{r}{x}$ gives

$$mt \ln\left(1 + \frac{r}{m}\right) = \left(\frac{r}{x}\right) t \ln(1 + x) = rt \left(\frac{\ln(1 + x)}{x}\right).$$

Since $\ln(1) = 0$, the quotient is the difference quotient for the natural logarithm at 1. Using the derivative studied in our final-year mathematics course, we obtain

$$\lim_{x \to 0} \frac{\ln(1 + x) - \ln(1)}{x} = \ln'(1) = \frac{1}{1} = 1.$$

Since rt is constant, it follows that

$$\lim_{m \to +\infty} \ln\left(\frac{A_m}{C_0}\right) = rt \times 1 = rt.$$

The exponential function is continuous on $\mathbb{R}$, so taking exponentials gives

$$\lim_{m \to +\infty} \frac{A_m}{C_0} = e^{rt} \implies \lim_{m \to +\infty} A_m = C_0 e^{rt}.$$

Taking logarithms thus makes the limit tractable and shows why continuous compounding follows an exponential model.

A.2 Why is $\ln(1+i) \approx i$ when i is small?

Question *You mentioned that* $\ln(1+i) \approx i$. *Can you justify it mathematically?*

Detailed answer This is a local linear approximation. Near a point, we approximate the curve by its tangent line, an idea studied in the final two years of French secondary school.

Let $f(x) = \ln(1+x)$. We want the equation of the tangent to its graph at $x = 0$. The general equation of the tangent at $x = a$ is

$$y = f'(a)(x-a) + f(a).$$

Here, $f'(x) = \frac{1}{1+x}$. Evaluating the function and its derivative at $a = 0$ gives

- $f(0) = \ln(1+0) = \ln(1) = 0$
- $f'(0) = \frac{1}{1+0} = 1$.

Substituting into the tangent line equation gives

$$y = 1 \times (x-0) + 0 \implies y = x.$$

Near $x = 0$, the graph of $\ln(1+x)$ and the line $y = x$ are very close. In finance, an annual interest rate i (or k) is often small when expressed as a decimal. For example, at $i = 3\% = 0.03$, the tangent gives $\ln(1+0.03) \approx 0.03$. The logarithm is $0.02955\ldots$. The absolute error is therefore about 4×10^{-4}.